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**NOTE ON  $q$ -DUAL THEORIES**

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**Abstract.** The  $q$ -deformation of the Lie algebras underlying the standard field theories leads to a pair of dual algebras. We describe a simple choice of possible field theories based on these derived algebras. One of these approximates the standard Lie theory of point particles, while the other is proposed as a field theory of knotted solitons.

## 1. Introduction.

In discussing  $q$ -groups underlying non-standard field theories we shall use the language of Lie groups rather than Hopf algebras, since we want to emphasize the correspondence limit with standard theories by terming the  $q$ -deformation of the standard Lie group an “internal algebra” and the  $q$ -deformation of the standard Lie algebra an “external algebra”. Matrix elements of the external algebra are numerically valued and are very close to the corresponding elements of the standard Lie algebra. On the other hand, matrix elements of the internal algebra are not numerically valued and have in general no finite matrix representation.

The external algebra, lying near the standard Lie algebra, may be made the basis of a point particle field theory approximating the standard point particle field theory. The internal algebra, carrying degrees of freedom present in neither the external algebra nor the corresponding point particle theory, may be regarded as the basis of a soliton theory. In the model here proposed these new degrees of freedom, implying non-locality, will describe solitons in the form of loops defined by the internal algebra.

We have previously discussed external  $q$ -gravity and  $q$ -electroweak and found that these theories are very close to the standard theories: Einstein gravity and Weinberg-Salam electroweak, but they are also incomplete.<sup>1,2</sup> Here we present a possible completion in the form of a solitonic version of the internal theory.

## 2. The Two Algebras.

The two-dimensional representation of  $SL_q(2)$  may be defined by

$$T^t \epsilon_q T = T \epsilon_q T^t = \epsilon_q \quad (2.1a)$$

where

$$\epsilon_q = \begin{pmatrix} 0 & q_1^{1/2} \\ -q^{1/2} & 0 \end{pmatrix} \quad q_1 = q^{-1} \quad (2.1b)$$

Then if

$$T = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad (2.2)$$

eq. (2.1) implies

$$\begin{aligned} ab &= qba & ac &= qca & bc &= cb & ad - qbc &= 1 \\ cd &= qdc & bd &= qdb & da - q_1 bc &= 1 \end{aligned} \quad (2.3)$$

If  $T$  is unitary,

$$\begin{aligned} c &= -q_1 \bar{b} \\ a &= \bar{d} \end{aligned} \quad (2.4)$$

The  $2j + 1$  dimensional representation of  $SU_q(2)$  is

$$D_{mm'}^j(a, \bar{a}, b, \bar{b}) = \Delta_{mm'}^j \sum_{s,t} \left\langle \begin{matrix} n_+ \\ s \end{matrix} \right\rangle_1 \left\langle \begin{matrix} n_- \\ t \end{matrix} \right\rangle_1 q^{t(n_++1-s)} (-)^t \times \delta(s+t, n'_+) a^s b^{n_+-s} \bar{b}^t \bar{a}^{n-t} \quad (2.5)$$

where

$$\begin{aligned} n_{\pm} &= j \pm m \\ n'_{\pm} &= j \pm m' \end{aligned} \quad \left\langle \begin{matrix} n \\ s \end{matrix} \right\rangle_1 = \frac{\langle n \rangle_1!}{\langle s \rangle_1! \langle n-s \rangle_1!} \quad \langle n \rangle_1 = \frac{q_1^{2n} - 1}{q_1^2 - 1}$$

and

$$\Delta_{mm'}^j = \left[ \frac{\langle n'_+ \rangle_1! \langle n'_- \rangle_1!}{\langle n_+ \rangle_1! \langle n_- \rangle_1!} \right]^{1/2}$$

Now set

$$D^{1/2}(a, \bar{a}, b, \bar{b}) = e^{B\sigma_+} e^{\lambda\theta\sigma_3} e^{C\sigma_-} \quad (2.6)$$

where  $q = e^\lambda$  and expand to terms linear in  $(B, C, \theta)$ . Then

$$D_{mm'}^j(a, \bar{a}, b, \bar{b}) = D_{mm'}^j(0, 0, 0) + B(J_B^j)_{mm'} + C(J_C^j)_{mm'} + 2\lambda\theta(J_\theta^j)_{mm'} + \dots \quad (2.7)$$

The non-vanishing matrix coefficients  $(J_B^j)_{mm'}$ ,  $(J_C^j)_{mm'}$  and  $(J_\theta^j)_{mm'}$  are by (2.5) and (2.7)

$$\begin{aligned} \langle m-1 | J_B^j | m \rangle &= [\langle j+m \rangle_1 \langle j-m+1 \rangle_1]^{1/2} \\ \langle m+1 | J_C^j | m \rangle &= [\langle j-m \rangle_1 \langle j+m+1 \rangle_1]^{1/2} \\ \langle m | J_\theta^j | m \rangle &= m \end{aligned} \quad (2.8)$$

Then  $(B, C, \theta)$  and  $(J_B, J_C, J_\theta)$  are the generators of the two dual algebras referred to in the introduction. They obey the following commutation rules:

**External algebra:**

$$(J_B, J_\theta) = -J_B \quad (J_C, J_\theta) = J_C \quad (J_B, J_C) = q_1^{2J-1} [2J_\theta] \quad (2.9)$$

where

$$[x] = \frac{q^x - q_1^x}{q - q_1} \quad \langle x \rangle = \frac{q^x - 1}{q - 1}$$

and

**Internal algebra:**

$$(B, C) = 0 \quad (\theta, B) = B \quad (\theta, C) = C \quad (2.10)$$

The matrix elements of the external algebra are given by (2.8) and (2.9). They are numerically valued and describe a deformation of the  $SU(2)$  algebra.

The arguments of  $D_{mm'}^j$ , either  $(B, C, \theta)$  or  $(a, \bar{a}, b, \bar{b})$  are the generators of the internal algebra and describe a deformation of the  $SU(2)$  group. They obey the algebra (2.10) or (2.3).

### 3. The Dual Actions of $SU_q(2)$

In the external theory the vector connection is expanded in the external algebra:

$$A_\mu(x) = \sum_{s=1}^3 A_\mu^s(x) J_s^1 \quad (3.1)$$

where  $J_s^1$  are matrices whose numerically valued matrix elements satisfy (2.8) with  $j = 1$ . The curvature is then

$$\begin{aligned} F_{\mu\lambda} &= \sum [(\partial_\mu A_\lambda^s - \partial_\lambda A_\mu^s) J_s^1 + A_\mu^s A_\lambda^t [J_s^1, J_t^1]] \\ &= \sum (\partial_\mu A_\lambda^p - \partial_\lambda A_\mu^p + f_{st}^p(q) A_\mu^s A_\lambda^t) J_p^1 \\ &= \sum F_{\mu\lambda}^p J_p^1 \end{aligned} \quad (3.2)$$

and the action is

$$S_{int} = -\frac{1}{4} \int d^4x \text{Tr} F_{\mu\lambda} F^{\mu\lambda} \quad (3.3)$$

$$\begin{aligned} &= -\frac{1}{4} \int d^4x \sum F_{\mu\lambda}^p F^{\mu\lambda n} \text{Tr} J_p^1 J_n^1 \\ &= -\frac{1}{4} \int d^4x \sum g_{pn}(q) F_{\mu\lambda}^p F^{\mu\lambda n} \end{aligned} \quad (3.4)$$

Here  $f_{st}^p$  and  $g_{pn}$  will depend on  $q$  since the matrix elements in (2.8) depend on  $q$  if  $j = 1$ .

In the internal theory the vector connection is expanded in the internal algebra.

$$A_\mu(a, \bar{a}, b, \bar{b}) = \sum_{jmn} A_\mu(x|jmn) D_{mn}^j(a, \bar{a}, b, \bar{b}) \quad (3.5)$$

where  $j$  is not restricted. The curvature is now

$$\begin{aligned} F_{\mu\lambda}(x|a, \bar{a}, b, \bar{b}) &= \sum_{jmn} [\partial_\mu A_\lambda(x|jmn) - \partial_\lambda A_\mu(x|jmn)] D_{jmn}(a, \bar{a}, b, \bar{b}) \\ &+ \sum_{\substack{j_1 m_1 n_1 \\ j_2 m_2 n_2}} A_\mu(x|j_1 m_1 n_1) A_\lambda(x|j_2 m_2 n_2) [D_{j_1 m_1 n_1}(a \bar{a} b \bar{b}), D_{j_2 m_2 n_2}(a \bar{a} b \bar{b})] \end{aligned} \quad (3.6)$$

To reduce the commutator expand the product of two  $D_{jmn}$  as follows:

$$D_{j_1 m_1 n_1} D_{j_2 m_2 n_2} = \sum_{jmn} \hat{C}_{j_1 m_1 n_1, j_2 m_2 n_2}^{jmn} D_{jmn} \quad (3.7)$$

Then

$$h(D_{j_1 m_1 n_1} D_{j_2 m_2 n_2} \bar{D}_{j_3 m_3 n_3}) = \sum_{jmn} \hat{C}_{j_1 m_1 n_1, j_2 m_2 n_2}^{jmn} h(D_{jmn} \bar{D}_{j_3 m_3 n_3}) \quad (3.8)$$

where  $h$  means a Woronowicz integration<sup>3</sup> over the  $q$ -group algebra and

$$h(D_{jmn} \bar{D}_{j_3 m_3 n_3}) = \delta_{jj_3} \delta_{mm_3} \delta_{nn_3} \frac{q^{-2n_3}}{[2j_3 + 1]_q} \quad (3.9)$$

Then

$$\hat{C}_{j_1 m_1 n_1, j_2 m_2 n_2}^{j_3 m_3 n_3} = q^{2n_3} [2j_3 + 1]_q h(D_{j_1 m_1 n_1} D_{j_2 m_2 n_2} \bar{D}_{j_3 m_3 n_3}) \quad (3.10)$$

The triple integral over the  $q$ -group algebra is essentially a  $q$ -Clebsch-Gordan coefficient.<sup>4</sup>

Hence

$$\begin{aligned} [D_{j_1 m_1 n_1}, D_{j_2 m_2 n_2}] &= \sum_{j_3 m_3 n_3} (\hat{C}_{j_1 m_1 n_1, j_2 m_2 n_2}^{j_3 m_3 n_3} - \hat{C}_{j_2 m_2 n_2, j_1 m_1 n_1}^{j_3 m_3 n_3}) D_{j_3 m_3 n_3} \\ &= \sum_{j_3 m_3 n_3} C_{j_1 m_1 n_1, j_2 m_2 n_2}^{j_3 m_3 n_3} D_{j_3 m_3 n_3} \end{aligned} \quad (3.11)$$

where

$$C_{12}^3 = \hat{C}_{12}^3 - \hat{C}_{21}^3 \quad (3.12)$$

Then

$$\begin{aligned} F_{\mu\lambda}(x|a\bar{a}b\bar{b}) &= \sum_{jmn} [\partial_\mu A_\lambda^{jmn}(x) - \partial_\lambda A_\mu^{jmn}(x) \\ &\quad + \sum_{\substack{j_1 m_1 n_1 \\ j_2 m_2 n_2}} C_{j_1 m_1 n_1, j_2 m_2 n_2}^{jmn} A_\mu^{j_1 m_1 n_1}(x) A_\lambda^{j_2 m_2 n_2}(x)] D_{jmn} \end{aligned} \quad (3.13)$$

This is necessarily an infinite component field. The sum in (3.13) is restricted only by  $j_1 + j_2 \geq j \geq |j_1 - j_2|$ .

The interior action is

$$S = -\frac{1}{4} h \int d^4x F_{\mu\lambda} F^{\mu\lambda} \quad (3.14)$$

$$\begin{aligned} &= -\frac{1}{4} \int d^4x F_{\mu\lambda}^{jmn} F^{\mu\lambda j' m' n'} h(D_{jmn} \bar{D}_{j' m' n'}) \\ &= -\frac{1}{4} \int d^4x \sum_{jmp} F_{\mu\lambda}^{jmp} F_{jmp}^{\mu\lambda} \frac{q^{-2p}}{[2j + 1]_q} \end{aligned} \quad (3.15)$$

by (3.9). Reduction of the action to a functional of  $c$ -number fields is accomplished in the external action by tracing and in the internal action by  $h$ -integration. To keep the correspondence with the external action we may restrict the sum in (3.15) to  $j = 1$ . Then

$$S_{int} = -\frac{1}{4} \int d^4x \sum_{1mp} F_{\mu\lambda}^{1mp} F_{1mp}^{\mu\lambda} \frac{q^{-2p}}{[3]_q} \quad (3.16)$$

The external and internal theories will differ in the two sets of structure constants, namely: on the one side the external  $f_{st}^p$  appearing in (3.2) that are very close to the corresponding Lie structure constants and on the other side the internal  $C_{j_1 m_1 n_1, j_2 m_2 n_2}^{jmn}$  appearing in (3.13) and dependent on the  $q$ -Clebsch-Gordan coefficients according to (3.11), (3.12) and (3.10). In addition, the curvature of the external theory is linear in the generators  $J_s$ , while the curvature of the internal theory is multinomial in the generators  $(a, \bar{a}, b, \bar{b})$ .

Both  $A_\mu^s(x)$  in (3.1) and  $A_\mu(x|jmn)$  in (3.5) are to be Fock expanded in states of momentum ( $p$ ) and polarization ( $r$ ). Point particles are to be associated with states  $(p, r, s)$  and solitons with states  $(p, r, jmn)$ .

#### 4. Interpretation of the Internal Theory.

The external algebra lying near the standard Lie algebra has been interpreted as the basis of a point particle field theory approximating the standard point particle field theory. The internal algebra carrying degrees of freedom not present in the external algebra, nor in the corresponding point particle theory, will now be interpreted as the basis of a soliton theory. One may also consider the possibility that the external theory arising from linearization of (2.5) in (2.7) is a linearized version of the internal theory. Since the internal  $q$ -theory is an infinite component theory, however, its linearized version also has infinitely many components. The external  $q$ -theory, which is confined to the adjoint representation and has the same number of components as the standard theory can therefore be related to only a truncated form of the internal  $q$ -theory.

The interpretation of the internal theory depends on the algebra of  $SU_q(2)$ . We are assuming that the soliton states are labelled by the irreducible representations of  $SU_q(2)$  as in Eq. (3.5). The  $D_{mn}^j(a, \bar{a}, b, \bar{b})$  appearing there are not numerically valued but are operators whose expectation value may be computed on the state space defined by the  $(a, \bar{a}, b, \bar{b})$  algebra. Since  $[b, \bar{b}] = 0$ , choose common eigenstates of  $b$  and  $\bar{b}$  as the basic states. Then  $\bar{a}$  and  $a$  are raising and lowering operators.

Assume

$$\begin{aligned} b|0\rangle &= \beta|0\rangle \\ \bar{b}|0\rangle &= \beta^*|0\rangle \end{aligned}$$

Then

$$\bar{b} \cdot \bar{a}^N |0\rangle = q^N \beta^* \cdot \bar{a}^N |0\rangle$$

or

$$\bar{b}|N\rangle = q^N \beta^* |N\rangle$$

There is an infinite number of states on which  $\langle N | D_{mn}^j(a\bar{a}b\bar{b}) | N \rangle$  exists.<sup>5</sup> In this sense there is an infinite number of degrees of freedom associated with each solitonic state  $D^j(m, n)$ . These expectation values will be polynomials in  $q, \beta$ , and  $\beta^*$ . The states of excitation,  $|N\rangle$ , of the soliton denoted by  $D^j(m, n)$  are analogous to the excited states of a string. Here the eigenvalues of  $b\bar{b}$  are arranged in geometric rather than arithmetic progression.

The solitons may alternatively be characterized by Jones or Kauffman knot polynomials that are defined by the  $q$ -algebra since  $\epsilon_q$ , the  $SU_q(2)$  invariant, encodes a program for these polynomials. To describe the Kauffman algorithm,<sup>6</sup> let  $X$  be any crossing in the graph  $K$ , representing an unoriented link, and let  $\langle K \rangle$  be the polynomial associated with  $K$ :

$$\langle K \rangle = \left\langle \begin{array}{c} \cdots \\ X \end{array} \right\rangle \quad (4.1)$$

First express the polynomial  $\langle K \rangle_n$  with  $n$  crossings as a linear combination of the polynomials  $\langle K_- \rangle_{n-1}$  and  $\langle K_+ \rangle_{n-1}$  each with  $n-1$  crossings:

$$\langle K \rangle_n = i \operatorname{Tr} \epsilon_q [\sigma_- \langle K_- \rangle_{n-1} + \sigma_+ \langle K_+ \rangle_{n-1}] \quad \sigma_{\pm} = \frac{1}{2}(\sigma_1 \pm i \sigma_2) \quad (4.2)$$

where  $K_-$  and  $K_+$  signify two ways of splicing away the crossing  $X$  by opening up either the - or + channels.<sup>6</sup>

Then iterating (4.2) to remove all crossings one obtains a linear combination of  $2^n$  polynomials and associated graphs, some with internal loops. The internal loops may then be removed by

$$\langle OK \rangle = (\operatorname{Tr} \epsilon_q \epsilon_q^t) \langle K \rangle \quad (4.3)$$

$$\langle O \rangle = \operatorname{Tr} \epsilon_q \epsilon_q^t \quad (4.4)$$

Application of these rules to a graph  $K$  with multiple crossings reduces  $\langle K \rangle$  to a Laurent polynomial in  $q$ . These rules relate  $\epsilon_q$ , the invariant of  $SU_q(2)$ , to the polynomials

characterizing the unoriented knot. If  $K$  is oriented, one may form the following invariant of ambient isotopy<sup>6</sup>

$$f_k(A) = (-A^3)^{-W(K)} \langle K \rangle \quad (4.5a)$$

where

$$A = i \text{ Tr } \epsilon_q \sigma_- \quad (4.5b)$$

and where  $W(K)$ , the sum of the crossing signs, is the writhe of  $K$ . For any oriented link

$$V_k(t) = f_k(t^{1/4}) \quad (4.6)$$

is the one variable Jones polynomial.<sup>6</sup>

It is then also possible to label a knot with the irreducible representation  $D_{\frac{w}{2} \frac{r+1}{2}}^{N/2}$  of  $SU_q(2)$ , where  $N$ ,  $w$ , and  $r$  signify the number of crossings, the writhe, and the rotation (the Whitney degree of the knot). Here  $N$ ,  $w$ , and  $r$  are all integers and  $w - r$  is required to be odd by a knot constraint. We now assume some of the normal modes of the vector potential in (3.5) represent knots labelled by  $D_{\frac{w}{2} \frac{r+1}{2}}^{N/2}$ , while the remaining normal modes represent either unknots (loops) or stringlike structures with the excited states of a “geometric” oscillator.

In the external  $q$ -electroweak theory the matrix elements in (2.8) and  $J_s^1$  in (3.1) are labelled by  $(j, m)$  where  $j$  is the isotopic spin ( $I$ ) and  $m$  is its  $z$ -component ( $I_3$ ). Then the indices labelling  $D_{\frac{w}{2} \frac{r+1}{2}}^{N/2}$  in (2.7) and (3.5) must be related to the isotopic spin if the external theory is regarded as a perturbative version of the internal theory, i.e.  $(N, w, r) \sim (I, I_3)$  and hypercharge.

The eigenvalues of the Hamiltonian corresponding to (3.15) would then depend on these quantum numbers as follows:

$$-\frac{1}{4} \int d\vec{x} [F_{ok}(N, w, r) F_{ok}(N, w, r) + F_{jk}(N, w, r) F_{jk}(N, w, r)] \frac{q^{-(r+1)}}{[N+1]}$$

There is then an additional fine structure depending on  $r$ .

A knot labelled by  $(N, w, r)$  and characterized by a Jones polynomial may be termed a classical knot. A knot labelled by  $(N, w, r)$ , but characterized by the operator  $D_{\frac{w}{2} \frac{r+1}{2}}^{N/2}(a, \bar{a}, b, \bar{b})$ , defines a state space and may be termed a quantum mechanical knot. Both the polynomial and the state space stem from  $SU_q(2)$ .

The physical picture suggested here arises from the substitution of  $SU_q(2)$  for  $SU(2)$  in our standard theories. New degrees of freedom are necessarily introduced by the  $q$ -formalism and they are naturally interpreted as solitonic. The identification of these solitons as knots is then suggested by the relation of knots to the  $SU_q(2)$  invariant  $\epsilon_q$ .



Knot states appear in attempts to quantize general relativity.<sup>7</sup> Since external  $q$ -gravity approximates standard gravity, we might expect knot states to appear in external  $q$ -gravity as well. Since the Kauffman rules are based on  $\epsilon_q$ , the invariant of the  $q$ -Lorentz group,  $SL_q(2)$ , the general argument of this section holds for internal  $q$ -gravity as well as for internal  $q$ -electroweak. Therefore knots may appear in both internal and external  $q$ -gravity. Since all fields are coupled to the gravitational field, knots may on these grounds be expected quite generally, and one may conjecture that  $SU_q(2)$  plays the role of a universal hidden symmetry.

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